

THE SCORZA CORRESPONDENCE IN GENUS 3

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ABSTRACT. In this note we prove the genus 3 case of a conjecture of G. Farkas and A. Verra on the limit of the Scorza correspondence for curves with a theta-null. Specifically, we show that the limit of the Scorza correspondence for a hyperelliptic genus 3 curve C is the union of the curve $\{x, \sigma(x) \mid x \in C\}$ (where σ is the hyperelliptic involution), and twice the diagonal. Our proof uses the geometry of the subsystem Γ_{00} of the linear system $|2\Theta|$, and Riemann identities for theta constants.

1. INTRODUCTION

Let \mathcal{M}_g be the moduli space of smooth complex curves of genus g , and let \mathcal{S}_g^\pm be the moduli spaces of smooth spin curves, i.e. the moduli of pairs consisting of $C \in \mathcal{M}_g$ and a line bundle η on C such that $\eta^{\otimes 2} = K_C$, and such that $h^0(C, \eta)$ is even (resp. odd) on \mathcal{S}_g^+ (resp. \mathcal{S}_g^-). Given a pair $(C, \eta) \in \mathcal{S}_g^+$ such that $h^0(C, \eta) = 0$, the *Scorza correspondence* is the curve in $C \times C$ defined by

$$S(C, \eta) := \{(x, y) \in C \times C \mid h^0(C, \eta + x - y) > 0\}.$$

The Scorza correspondence is a classical construction in algebraic geometry, a beautiful modern exposition, many further details, and references for which are given in [Dol10, section 5.5].

Since $\eta^{\otimes 2} = K_C$, it follows from Riemann-Roch theorem that the Scorza correspondence is symmetric, and can be considered in $\text{Sym}^2 C$. Moreover, as η was assumed to be non-effective, the correspondence S cannot intersect the diagonal in $C \times C$.

Considered globally over \mathcal{S}_g^+ , the Scorza correspondences form a codimension one family in $\mathcal{S}_{g,2}^+$ (the moduli of spin curves with two distinct marked points), defined away from the locus where η is effective. This locus is called the theta-null divisor

$$\theta_{\text{null}} := \{(C, \eta) \in \mathcal{S}_g^+ \mid h^0(C, \eta) > 0\},$$

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and plays an important role in the study of the birational geometry of the moduli of spin curves by Farkas and Verra [Far10, FV09]. It is non-empty for $g \geq 3$.

Naturally one can take the closure of the universal Scorza correspondence in $\mathcal{S}_{g,2}^+$, and study the fibers over the theta-null divisor, i.e. the strict transform of the Scorza correspondence as the curve acquires a theta-null. While in general not much is known about the geometry of curves with a theta-null, a test case is provided by hyperelliptic curves (which are in fact characterized by the vanishing of a certain collection of theta-nulls [Mum07, Poo94]).

The following conjecture, due to Farkas and Verra, was communicated to us by Gavril Farkas:

Conjecture. *For a generic hyperelliptic curve C with the hyperelliptic involution σ , and a vanishing even theta characteristic η on C , the limit of the Scorza correspondence in $C \times C$ scheme-theoretically is the union of the curve $\{x \times \sigma(x) \mid x \in C\}$, and of the diagonal with multiplicity two.*

Our main result, theorem 10, is the proof of this conjecture for the case of $g = 3$, for all hyperelliptic curves of genus 3 — in which case each hyperelliptic curve has precisely one vanishing theta-null.

Another motivation for studying the Scorza correspondence is the problem of constructing complete subvarieties of \mathcal{M}_g . Recall that by a theorem of Diaz [Dia84] (see also [GK10] for a different new approach) there do not exist complete subvarieties of \mathcal{M}_g of dimension larger than $g - 2$, while all known explicit examples of complete subvarieties are of dimension equal to a constant multiple of $\ln g$ (for $g \gg 0$) (see [HM98] for more details).

A common theme in constructing such complete subvarieties is by using an appropriate cover construction. The starting point for such constructions is the complete curve $X \subset \mathcal{M}_3$ constructed explicitly by Zaal [Zaa95] (the existence of such a curve follows a priori from the existence of the projective Satake compactification \mathcal{M}_3^{Sat} such that the boundary $\partial \mathcal{M}_3^{Sat}$ is codimension two, so that cutting $\mathcal{M}_3^{Sat} \subset \mathbb{P}^N$ by sufficiently general hypersurfaces yields a curve that can be made to avoid $\partial \mathcal{M}_3^{Sat}$ — and which is thus contained in \mathcal{M}_3). By considering a suitable cover of X one thus obtains a complete surface in \mathcal{M}_7 , and this can be iterated further.

Alternatively, one could try to use the Scorza correspondence (this idea is due to Gavril Farkas): indeed, consider the preimage of X in \mathcal{S}_3^+ , and consider the universal Scorza correspondence S over it. For any $(C, p, q) \in S \subset \mathcal{S}_{3,2}^+$ if p and q are distinct, the double cover of C

branching at p and q is a smooth curve of genus 6. Thus if the Scorza correspondence were to never intersect the diagonal of $C \times C$ (including over the hyperelliptic locus), by varying $C \in X$ and $p, q \in S$ one would get a complete surface in \mathcal{M}_6 . Thus our main result shows that this cannot be made to work, as indeed the complement of the hyperelliptic locus $\mathcal{M}_3 \setminus \mathcal{H}_3$ is affine and does not contain any complete curves. Note that in characteristic $p > 2$ a complete surface in \mathcal{M}_6 was constructed by Zaal in [Zaa99].

The method of our proof is by using the difference map $C \times C \rightarrow C - C \subset J(C)$, and using the explicit description of the image as the base locus of the linear subsystem $\Gamma_{00} \subset |2\Theta|$ on the Jacobian. Using the explicit knowledge of this linear system and suitable identities for theta constants then yield the defining equations for the strict transform of the Scorza correspondence over the hyperelliptic curves, which allows us to prove the main result.

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2. THETA CONSTANTS AS MODULAR FORMS

In this section we gather some definitions and results about theta constants and the rings of modular forms.

We denote by \mathbb{H}_g the Siegel upper half-space consisting of $g \times g$ complex symmetric matrices τ with positive-definite imaginary part. For $\tau \in \mathbb{H}_g$ we denote by $A_\tau := \mathbb{C}^g / \mathbb{Z}^g \tau + \mathbb{Z}^g$ the abelian variety corresponding to τ . We denote by

$$\theta(\tau, z) := \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \tau n + 2\pi i n^t z)$$

the Riemann theta function $\theta : \mathbb{H}_g \times \mathbb{C}^g \rightarrow \mathbb{C}$. The zero locus in z of the theta function is invariant under translating z by $\mathbb{Z}^g \tau + \mathbb{Z}^g$, and thus defines the theta divisor $\Theta_\tau \subset A_\tau$, which gives a principal polarization on A_τ .

For $\varepsilon, \delta \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ (which we think of as a two-torsion point $m = (\varepsilon\tau + \delta)/2$ on A_τ) we denote by $\theta_m(\tau, z) := \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z)$ the theta function with characteristics — which, up to an easy exponential factor, is the same as $\theta(\tau, z + m)$. It is known that $h^0(A_\tau, 2\Theta_\tau) = 2^g$ for any $\tau \in \mathbb{H}_g$.

and that the basis for sections is given by theta functions of the second order: for $\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^g$ we denote

$$\Theta[\varepsilon](\tau, z) := \theta \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} (2\tau, 2z) = \sum_{n \in \mathbb{Z}^g} \exp(2\pi i (n + \varepsilon/2)^t (\tau(n + \varepsilon/2) + 2z))$$

Since for any two-torsion point $m \in A_\tau[2]$ the square $\theta_m^2(\tau, z)$ is a section of $2\Theta_\tau$ (the shift $2m$ is zero), it is a linear combination of theta functions of the second order, given explicitly by Riemann's bilinear addition formula:

$$(1) \quad \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, z)^2 = \sum_{\sigma \in (\mathbb{Z}/2\mathbb{Z})^g} (-1)^{\sigma \cdot \delta} \Theta[\sigma](\tau, z) \Theta[\sigma + \varepsilon](\tau, 0).$$

The symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on \mathbb{H}_g by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ \tau = (A\tau + B)(C\tau + D)^{-1},$$

where we represent an element $\gamma \in \mathrm{Sp}(2g, \mathbb{Z})$ as four $g \times g$ blocks. The quotient under this action is the moduli space of principally polarized abelian varieties (ppav) $\mathcal{A}_g = \mathbb{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$. For future use we also denote $\mathcal{U}_g \rightarrow \mathcal{A}_g$ the universal family of ppav over \mathcal{A}_g (which is a quotient of $\mathbb{H}_g \times \mathbb{C}^g$ under a suitable action of $\mathrm{Sp}(2g, \mathbb{Z}) \ltimes \mathbb{Z}^{2g}$).

Definition 1. For a finite index subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$ and an integer r , a multiplier system of weight $r/2$ is a map $v : \Gamma \rightarrow \mathbb{C}^*$, such that the map

$$\gamma \mapsto v(\gamma) \det(C\tau + D)^{r/2}$$

satisfies the cocycle condition for every $\sigma \in \Gamma$ and $\tau \in \mathbb{H}_g$ (note that the function $\det(C\tau + D)$ possesses a square root).

Clearly a multiplier system of integral weight is a character.

Definition 2. A function $f : \mathbb{H}_g \rightarrow \mathbb{C}$ is called a modular form of weight $r/2$ with multiplier v , with respect to a finite index subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$ if

$$(2) \quad f(\gamma \cdot \tau) = v(\gamma) \det(C\tau + D)^{r/2} f(\tau) \quad \forall \tau \in \mathbb{H}_g, \forall \gamma \in \Gamma,$$

and if additionally f is holomorphic at all cusps of \mathbb{H}_g/Γ .

We denote by $[\Gamma, r/2, v]$ the finite dimensional space of modular forms. We denote L the bundle of modular forms of weight 1 on \mathcal{A}_g , so that by definition a modular form of weight r (with trivial multiplier) with respect to the entire group $\mathrm{Sp}(2g, \mathbb{Z})$ is a section of $L^{\otimes r}$: so we have $[\mathrm{Sp}(2g, \mathbb{Z}), r, 1] = H^0(\mathcal{A}_g, L^{\otimes r})$.

The restrictions of theta functions with characteristics (resp. theta functions of the second order) to $z = 0$ are called theta constants with characteristics (resp. of the second order). It is known that $\theta_m(\tau, 0)$ are modular forms of weight one half with respect to a certain finite index normal subgroup $\Gamma_g(4, 8) \subset \mathrm{Sp}(2g, \mathbb{Z})$ (the action of $\Gamma_g(2)$ adds certain fourth roots of unity, while the action of full $\mathrm{Sp}(2g, \mathbb{Z})$ adds certain complicated eighth roots of unity and permutes characteristics under an affine action), while theta constants of the second order are modular forms, also of weight one half, with respect to a bigger subgroup $\Gamma_g(2, 4)$. Using the same method as in [SMT93], we can prove that the commutator of the group $\Gamma_g(2, 4)$ is the group $\Gamma_g(2, 4, 8)$. Moreover its characters appear in the transformation formula of theta constants, see [SM94a]. We denote $\mathcal{A}_g(2, 4) := \mathbb{H}_g / \Gamma_g(2, 4)$ the finite Galois cover of \mathcal{A}_g corresponding to the level $(2, 4)$ subgroup. Theta constants of the second order are known to define a morphism

$$Th : \mathcal{A}_g(2, 4) \rightarrow \mathbb{P}^{2^g-1}; \quad \tau \mapsto \{\Theta[\varepsilon](\tau, 0)\}_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^g},$$

that is generically 1-to-1, finite, and is known to be an embedding when $g \leq 3$ — see [Run93, SM94b] (the similar map of $\mathcal{A}_g(4, 8)$ given by theta constants with characteristics is known to be an embedding for all g). Letting v be the multiplier of the theta constants of the second order, we set

$$M(\Gamma_g(2, 4), v) := \bigoplus_{r=0}^{\infty} [\Gamma_g(2, 4), r/2, v^r].$$

This is an integrally closed ring; moreover, for $g \leq 3$

$$M(\Gamma_g(2, 4), v) = \mathbb{C}[\Theta[\sigma](\tau, 0)],$$

i.e. theta constants of the second order generate this ring of modular forms. We refer to [Igu72, Run93, SM94a] for more details on theta constants and associated maps.

3. THE LINEAR SYSTEM Γ_{00}

In this section we recall the definition of the linear system Γ_{00} introduced by van Geemen and van der Geer [vGvdG86], and some results about it.

For a curve $C \in \mathcal{M}_g$ let $A : C \hookrightarrow J(C)$ be the Abel-Jacobi embedding of the curve into its Jacobian. Let $C - C \subset J(C)$ denote the (singular) surface that is the image of the map $C \times C \rightarrow J(C)$ given by $(p, q) \mapsto A(p) - A(q)$ (which blows down the diagonal). Recall that the degree of the map $C \times C \rightarrow C - C$ is equal to 1 for non-hyperelliptic curves, and 2 for hyperelliptic curves.

For any ppav A_τ the linear subsystem $\Gamma_{00} \subset |2\Theta_\tau|$ is defined by van Geemen and van der Geer [vGvdG86] to consist of those sections vanishing at the origin $z = 0$ to order at least 4. For a Jacobian, Γ_{00} coincides with the vector subspace of $H^0(A_\tau, 2\Theta_\tau)$ consisting of sections vanishing along $C - C$, see [vGvdG86]. Welters [Wel86] showed that as a set the base locus of Γ_{00} on a Jacobian is equal to $C - C$ (for $g = 4$ together with two exceptional points corresponding to the g_1^3 's on the curve). Izadi [Iza91] showed that the base locus of Γ_{00} on a non-hyperelliptic Jacobian is equal to $C - C$ as a scheme.

For any indecomposable ppav A the dimension of Γ_{00} is equal to $2^g - g(g+1)/2 - 1$, and a set of generators for Γ_{00} is constructed in [vGvdG86] in the following way (see [GSM10] for a different approach). Let $I \in I(\overline{Th(\mathcal{A}_g(2,4))})$ be a homogenous polynomial in theta constants of the second order vanishing identically. van Geemen and van der Geer [vGvdG86] show (using the heat equation for the theta function) that the expression

$$(3) \quad F_I := \sum_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^g} \frac{\partial I}{\partial \Theta[\varepsilon](\tau, 0)} \Theta[\varepsilon](\tau, z)$$

then gives an element in Γ_{00} , and moreover they prove that such F_I for all I in the ideal generate the linear system Γ_{00} for any indecomposable ppav.

4. THE SURFACE $C - C$ IN GENUS 3

In this section we specialize the preceding discussion to write down in genus 3 the defining equation for the universal family of surfaces $C - C \subset \mathcal{U}_3$.

In genus 3 the image $\overline{Th(\mathcal{A}_3(2,4))}$ is a hypersurface in \mathbb{P}^7 . The equation for this hypersurface, of degree 16, was first obtained by Schottky (perhaps Frobenius), and we quote the formula from [vGvdG86]. For the degree four polynomials in theta constants defined as

$$r_1 := \theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$r_2 := \theta \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \theta \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \theta \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \theta \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$r_3 := \theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

the Riemann quartic addition theorem gives $r_1 - r_2 - r_3 = 0$, which implies [vGvdG86, formula (7)]

$$(4) \quad I_3(\tau) := r_1^2 + r_2^2 + r_3^2 - 2r_1^2 r_2^2 - 2r_1^2 r_3^2 - 2r_2^2 r_3^2 = 0.$$

Using Riemann's bilinear addition formula (1) one can rewrite I_3 as a polynomial of degree 16 in theta constants of the second order $\Theta[\sigma](\tau)$. It then turns out that this polynomial on \mathbb{P}^7 is in fact independent of the choice of a syzygetic subspace used to define r_i , and is the one defining equation for $\overline{Th(\mathcal{A}_3(2, 4))} \subset \mathbb{P}^7$.

An equivalent expression for $I_3(\tau)$, manifestly invariant under the action of $\mathrm{Sp}(2g, \mathbb{Z})$, was obtained by Igusa [Igu81], who showed that up to a constant factor

$$I_3(\tau) = 2^3 \sum_{\varepsilon, \delta \in (\mathbb{Z}/2\mathbb{Z})^g} \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}^{16}(\tau, 0) - \left(\sum_{\varepsilon, \delta \in (\mathbb{Z}/2\mathbb{Z})^g} \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}^8(\tau, 0) \right)^2.$$

Thus from the results of van Geemen and van der Geer it follows that for any indecomposable ppav A_τ of dimension 3 the one-dimensional linear system Γ_{00} is generated by the function

$$F(\tau, z) := F_{I_3}(\tau, z).$$

To write F explicitly, note from [vGvdG86] that instead of taking partial derivatives in (3) it is possible to take partial derivatives with respect to squares of theta constants with characteristics, and then sum over all even characteristics, i.e. up to a constant factor we have

$$(5) \quad F(\tau, z) = \sum_{\varepsilon, \delta \in (\mathbb{Z}/2\mathbb{Z})^6} \frac{\partial I_3}{\partial \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, 0)^2} \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z)^2$$

and using formula (4) this can be written out explicitly.

5. THE SCORZA CORRESPONDENCE IN GENUS 3

In this section we describe analytically the (universal) Scorza correspondence in genus 3.

Recall that the Scorza correspondence for $(C, \eta) \in \mathcal{S}_g^+$ for η non-effective is the set of pairs of points $(x, y) \in C \times C$ such that $\eta + x - y$ is an effective divisor. The choice of a spin structure η fixes a choice of a symmetric principal polarization Θ_η on the Jacobian $J(C)$ such that $\Theta_\eta|_{A(C)} = \eta$. The image of the Scorza correspondence under the map $C \times C \longrightarrow C - C \subset J(C)$ is then the set of points $x - y \in C - C$ such that $\theta_\eta(\tau, x - y) = 0$, where τ is the period matrix of C . Thus the image of the Scorza correspondence in $J(C)$ is the intersection of

$C - C$ with the theta divisor. Using the above description of $C - C$ as the zero locus of $F(\tau, z)$ analytically we can write down the Scorza curve in $A_\tau = J(C)$ for C non-hyperelliptic as

$$(6) \quad S_\tau := (C - C) \cap \Theta_\tau = \{z \in A_\tau \mid F(\tau, z) = \theta_\eta(\tau, z) = 0\}$$

In what follows, working analytically over the Siegel space, to simplify notations we will choose the theta characteristic to be zero (since the deck transformations of the cover $\mathcal{S}_g^+ \rightarrow \mathcal{M}_g$ act transitively on the set of η , this will be enough).

Note that we can define the Scorza variety $S_\tau \subset A_\tau$ by the same formula for any $\tau \in \mathbb{H}_3$, even if the corresponding curve is reducible or hyperelliptic. We then see that S_τ is a curve unless the irreducible divisor $\Theta_\tau \subset A_\tau$ coincides with the surface $C - C$, which we know is given in A_τ by the equation $\{F(\tau, z) = 0\}$. Since $0 \in C - C$, for this to happen we must have in particular $0 \in \Theta_\tau$, i.e. we must have $\theta(\tau, 0) = 0$. Thus this happens precisely on the hyperelliptic locus (with the appropriate choice of the theta characteristic, i.e. on $\theta_{\text{null}} \subset \mathcal{S}_3^+$). In this case $\theta(\tau, z)$ is an even function of z vanishing at 0, so it vanishes to the second order at $z = 0$, its square vanishes to the fourth order, and thus $\theta^2(\tau, z)$ is the non-zero generator of Γ_{00} . We thus proved the following

Lemma 3. *On the hyperelliptic locus given by $\theta(\tau, 0) = 0$ the function F vanishes on the theta divisor: this is to say that we have*

$$\theta(\tau, 0) = \theta(\tau, z) = 0 \implies F(\tau, z) = 0.$$

6. AN EXPRESSION FOR $F(\tau, z)$

We now consider the embedding of the universal Kummer variety of dimension 3:

$$UTh : \mathcal{U}_3(2, 4) / \pm 1 \hookrightarrow \mathbb{P}^7 \times \mathbb{P}^7$$

given by

$$(\tau, z) \mapsto \{\Theta[\varepsilon](\tau, 0)\}_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^g} \times \{\Theta[\varepsilon](\tau, z)\}_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^g}$$

The defining ideal \mathcal{I} for the image of UTh is known quite explicitly. Indeed, let p_1 and p_2 be the two projections from $\mathbb{P}^7 \times \mathbb{P}^7$ onto the factors. Then the map $p_1 \circ UTh$ factors through $\mathcal{A}_3(2, 4)$, where it is simply Th , and thus I_3 is the single defining equation for the closure of $p_1 \circ UTh(\mathcal{U}_3(2, 4))$ in \mathbb{P}^7 .

On the other hand, for a fixed abelian variety (which, since $Th = p_1 \circ UTh$ is an embedding, means for $p_1 \circ UTh$ fixed), the map $p_2 \circ UTh$ is the map $|2\Theta_\tau| : A_\tau / \pm 1 \hookrightarrow \mathbb{P}^7$, equations for which are known by work of Mumford [Mum66, Mum67, Mum67].

In fact these equations are polynomial equations for $\Theta[\sigma](\tau, z)$ with coefficients themselves being polynomials in $\Theta[\varepsilon](\tau, 0)$ — thus they are explicit polynomials on $\mathbb{P}^7 \times \mathbb{P}^7$.

The above lemma then yields the following

Proposition 4. *The function $F(\tau, z)$ considered as a bihomogenous polynomial (of bidegree $(15, 1)$) on $\mathbb{P}^7 \times \mathbb{P}^7$, lies in the radical of the ideal generated by \mathcal{I} and the functions $\theta(\tau, 0)$ and $\theta^2(\tau, z)$.*

This proposition is of course simply a reformulation of the lemma; note that while by Riemann's bilinear addition formula $\theta^2(\tau, 0)$ is a polynomial in theta constants of the second order, $\theta(\tau, 0)$ itself is not.

We will now determine explicitly an expression for F in terms of the generators of this ideal, using the explicit formula (5) for F . Let $V_i \subset (\mathbb{Z}/2\mathbb{Z})^6$ for $i = 1, 2, 3$ be the set of four theta characteristics appearing in the product r_i . We need to compute the partial derivatives of (4) with respect to various theta constants. Obviously for $m \notin (V_1 \sqcup V_2 \sqcup V_3)$ such a partial derivative is zero, while for $m \in V_i$ we have

$$\frac{\partial I_3}{\partial \theta_m^2(\tau, 0)} = 2 \left(2r_i^2 - \sum_{j=1}^3 r_j^2 \right) \frac{r_i^2}{\theta_m^2(\tau, 0)}.$$

Summing over all m yields

$$F(\tau, z) = \sum_{i=1}^3 2 \left(2r_i^2 - \sum_{j=1}^3 r_j^2 \right) \sum_{m \in V_i} \frac{r_i^2}{\theta_m^2(\tau, 0)} \theta_m^2(\tau, z)$$

We want to evaluate $F(\tau, z)$ on the hyperelliptic locus, which is to say on $\theta_{\text{null}} \subset \mathcal{S}_3^+$. Lifting to \mathbb{H}_3 it means we are working on the locus of τ such that $\theta(\tau, 0) = \theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\tau, 0) = 0$. In this case of course we have $r_1 = 0$, which implies $r_2 + r_3 = 0$ and thus $r_2^2 - r_3^2 = (r_2 - r_3)(r_2 + r_3) = 0$. Since this product vanishes on the locus $\theta(\tau, 0)$, it implies that the modular form $r_2^2 - r_3^2$ is divisible by $\theta(\tau, 0)$.

Remark 5. For future use we note that $r_2^2 - r_3^2$ is not divisible by $\theta(\tau, 0)^2$: indeed, we know that $r_2 + r_3 = r_1$ is divisible by $\theta(\tau, 0)$. If $r_2 - r_3$ were also divisible by r_1 , this would imply that r_2 and r_3 were both divisible by r_1 , which is of course false as zero loci of different theta constants are different. On the other hand, if $r_2 + r_3 = r_1$ were divisible by $\theta(\tau, 0)^2$, then r_1 would be divisible by $\theta(\tau, 0)^2$, which is also not the case.

This computation allows us to write F explicitly. To simplify the formulas, we introduce notation for various expressions appearing in

the formulas for the derivatives of I with respect to various $\theta_m^2(\tau, 0)$. To this end, for any $m \in V_i$, $i = 2, 3$ we set

$$A_m(\tau) := (-1)^i 2(r_2 - r_3) \frac{r_1}{\theta(\tau, 0)} \cdot \frac{r_i^2}{\theta_m^2(\tau, 0)};$$

for any $m \in V_1 \setminus 0$ we set

$$B_m(\tau) := 2(r_1^2 - r_2^2 - r_3^2) \frac{r_1^2}{\theta^2(\tau, 0) \theta_m^2(\tau, 0)},$$

and finally we set

$$C(\tau) := 2(r_1^2 - r_2^2 - r_3^2) \frac{r_1^2}{\theta^2(\tau, 0)}.$$

In this notation we then have by adding all the partial derivatives of I

$$(7) \quad \begin{aligned} F(\tau, z) = & \theta(\tau, 0) \sum_{m \in V_2 \sqcup V_3} A_m(\tau) \theta_m^2(\tau, z) \\ & + \theta^2(\tau, 0) \sum_{m \in V_1 \setminus 0} B_m(\tau) \theta_m^2(\tau, z) + C(\tau) \theta^2(\tau, z) \end{aligned}$$

with $A_m(\tau)$ not divisible by $\theta(\tau, 0)$. Denoting the first two sums $A(\tau, z)$ and $B(\tau, z)$ respectively, we can write this as

$$(8) \quad F(\tau, z) = \theta(\tau, 0) A(\tau, z) + \theta^2(\tau, 0) B(\tau, z) + C(\tau) \theta^2(\tau, z).$$

Recall that $F(\tau, z) = 0$ within $A_\tau = J(C)$ is the defining equation for the surface $C - C \subset J(C)$, which in particular contains zero, so that we must have

$$0 = \theta(\tau, 0) A(\tau, 0) + \theta^2(\tau, 0) (B(\tau, 0) + C(\tau))$$

for all τ , which implies that

$$A(\tau, 0) = -\theta(\tau, 0) (B(\tau, 0) + C(\tau))$$

for all τ , and in particular we get

Lemma 6. *The expression $A(\tau, 0)$ defined above vanishes on the component of the hyperelliptic locus where $\theta(\tau, 0) = 0$: we have $\theta(\tau, 0) = 0 \implies A(\tau, 0) = 0$.*

Remark 7. Note that we could have formulas (7) and (8) satisfied with other $A_m(\tau), B_m(\tau), C(\tau)$, as the theta functions $\theta_m^2(\tau, z)$ appearing there are not linearly independent. However, the conclusion above about the vanishing of $A(\tau, 0)$ on the hyperelliptic locus is independent of this assumption.

7. THE SCORZA CORRESPONDENCE OVER THE HYPERELLIPTIC LOCUS IN GENUS 3

Recall that the Scorza correspondence within A_τ is given by the equations

$$S_\tau = \{\theta(\tau, z) = F(\tau, z) = 0\}.$$

Substituting here the expression for $F(\tau, z)$ from formula (8) and noticing that the term with $C(\tau)$ vanishes on the theta divisor we get equivalently

$$S_\tau = \{0 = \theta(\tau, z) = \theta(\tau, 0)(A(\tau, z) + \theta(\tau, 0)B(\tau, z))\}.$$

Away from the component of the hyperelliptic locus where $\theta(\tau, 0) = 0$ we can thus take out that factor from the second equation. Thus the limit of the Scorza correspondence (i.e. the intersection of the closure of the family of $S_\tau \subset \mathcal{U}_3$ with the fiber) as τ approaches the period matrix τ_0 of a hyperelliptic curve $C_0 \in \mathcal{H}_3$ such that $\theta(\tau_0, 0) = 0$ is given by

$$(9) \quad S_{\tau_0} = \{0 = \theta(\tau_0, z) = A(\tau_0, z)\}.$$

We note that if $A(\tau, z)$ were divisible by $\theta(\tau, 0)$, then in the defining equations for S_τ we could take out the factor of $\theta^2(\tau, 0)$, and thus the formula for S_{τ_0} would be different: it would involve $A(\tau, z)/\theta(\tau, 0)$ instead of $A(\tau, 0)$.

We will now prove that this is not the case:

Proposition 8. *The expression $A(\tau, z)$ is not divisible by $\theta(\tau, 0)$: there does not exist a holomorphic function $D(\tau, z)$ — which would then be a modular function — such that $A(\tau, z) = D(\tau, z)\theta(\tau, 0)$. Equivalently, there does not exist a holomorphic D such that*

$$(10) \quad F(\tau, z) = \theta^2(\tau, 0)D(\tau, z) + C(\tau)\theta^2(\tau, z)$$

Proof. Suppose for contradiction that we have (10) valid with some holomorphic function $D(\tau, z)$. Since $C(\tau) \in A(\Gamma_3(2, 4), v)$, it is a polynomial in the theta constants of the second order. Furthermore, $D(\tau, z)$ for fixed τ as a function of z is a section of $2\Theta_\tau$ (as are the other terms in (10)). Since theta functions of the second order form a basis of $H^0(A_\tau, 2\Theta_\tau)$, we must then have

$$D(\tau, z) = \sum_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^3} D_\varepsilon(\tau)\Theta[\varepsilon](\tau, z)$$

with $D_\varepsilon(\tau)$ some modular forms of weight $\frac{13}{2}$ wrt $\Gamma_3(2, 4)$, and hence each D_ε a polynomial of degree 13 in the theta constants of the second order.

Using Riemann's bilinear addition theorem for $\theta^2(\tau, z)$ we would then get from (10) the identity

$$\begin{aligned} F(\tau, z) &= \sum_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^3} (\theta^2(\tau, 0)D_\varepsilon(\tau) + C(\tau)\Theta[\varepsilon](\tau, 0))\Theta[\varepsilon](\tau, z) \\ &= \sum_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^3} F_\varepsilon(\tau)\Theta[\varepsilon](\tau, z) \end{aligned}$$

with $F_\varepsilon(\tau)$ a polynomial of degree 15 in the $\Theta[\varepsilon](\tau, 0)$.

Using Maple we verified that in the polynomial ring generated by theta functions and constants of the second order, by $\Theta[\varepsilon](\tau, z)$ and $\Theta[\varepsilon](\tau, 0)$, the polynomial $F(\tau, z)$ does not lie in the ideal generated by $\theta^2(\tau, 0)$ and $\theta^2(\tau, z)$, and thus the above expression is impossible. \square

We will now investigate the resulting limit of the Scorza correspondence, given by formula (9). We compute by definition

$$A(\tau_0, z) = \frac{2r_1(\tau_0)(r_2(\tau_0) - r_3(\tau_0))}{\theta(\tau_0, 0)} \left(r_2^2 \sum_{m \in V_2} \frac{\theta_m^2(\tau_0, z)}{\theta_m^2(\tau_0, 0)} - r_3^2 \sum_{m \in V_3} \frac{\theta_m^2(\tau_0, z)}{\theta_m^2(\tau_0, 0)} \right),$$

which using $0 = r_1(\tau_0) = r_2(\tau_0) + r_3(\tau_0)$ implies that its vanishing locus in z is equal to that of

$$H(\tau_0, z) := r_2^2(\tau_0) \sum_{m \in V_2} \frac{\theta_m^2(\tau_0, z)}{\theta_m^2(\tau_0, 0)} - r_3^2(\tau_0) \sum_{m \in V_3} \frac{\theta_m^2(\tau_0, z)}{\theta_m^2(\tau_0, 0)}.$$

Recall that by a conjecture of Farkas and Verra we expect to have $S_{\tau_0} = \{x - \sigma(x) | x \in C_0\}$, where σ denotes the hyperelliptic involution on C_0 . We first prove the inclusion.

Proposition 9. *The function $H(\tau_0, z)$ vanishes identically for $z = x - \sigma(x)$, and thus $S_{\tau_0} \supset \{x - \sigma(x) | x \in C_0\}$.*

Proof. We choose an Abel-Jacobi embedding of C_0 into its Jacobian in such a way that the involution lifts to -1 , so that we need to check that $H(\tau_0, 2x) = 0$ for any x on the curve. As a function of z , $H(\tau, z)$ is a section of $2\Theta_\tau$; hence $H(\tau, 2z)$ is a section of $8\Theta_\tau$. Since the theta function has degree 3 on a curve of genus 3, $H(\tau_0, 2x)$ is then a section of a line bundle of degree 24 on C_0 . Thus if we show it has more than 24 zeroes (counted with multiplicity), it is identically zero.

Note that by definition the 8 Weierstrass points are the fixed points of the involution σ , so if x is a Weierstrass point, we have $H(\tau_0, x - \sigma(x)) = H(\tau_0, 0) = 0$.

Since we are in genus 3 we can choose the Abel-Jacobi map of C_0 in such a way that the eight Weierstrass point map to some points $x_0 =$

$0, x_1, \dots, x_7$ of the Jacobian $J(C_0)$, and the associated characteristics are 0 and m_1, \dots, m_7 with all m_i odd. A consequence of this choice is $\theta(\tau_0, 0) = 0$. So our choice of Abel-Jacobi map is consistent with our hypothesis.

Since $H(\tau_0, z)$ is an even function of z , its gradient vanishes at each two-torsion point on $J(C_0)$, and thus automatically H vanishes at each Weierstrass point to order at least 2. So we get 16 zeroes, counted with multiplicity. We now show that H vanishes at each Weierstrass point to order at least 4, so the total number of zeroes is at least 32 (and thus $H(\tau_0, 2x)$ will vanish identically). To achieve this we will check that the second derivative is zero (the third derivative is zero by parity).

Indeed, we need to check that $\partial_U \partial_U H(\tau_0, z)|_{z=0} = 0$, where U denotes the tangent vector to the curve $C_0 \subset J(C_0)$ at the Weierstrass point (which, without loss of generality, we can take to be 0). Note that we have $\theta(\tau_0, 0) = 0$, so that $r_1(\tau_0) = 0$ and thus $r_2(\tau_0) + r_3(\tau_0) = 0$. By the heat equation, and using the parity of each θ_m as a function of z , we then get, up to a constant factor,

$$\begin{aligned} \partial_U \partial_U H(\tau_0, z)|_{z=0} &= 2r_2^2(\tau_0) \sum_{m \in V_2} \frac{\partial_{U \otimes U} \theta_m(\tau_0, 0)}{\theta_m(\tau_0, 0)} - 2r_3^2(\tau_0) \sum_{m \in V_3} \frac{\partial_{U \otimes U} \theta_m(\tau_0, 0)}{\theta_m(\tau_0, 0)} \\ &= \partial_{U \otimes U} (r_2^2(\tau_0) - r_3^2(\tau_0)) = 2r_2(\tau_0) \partial_{U \otimes U} (r_2(\tau_0) + r_3(\tau_0)) \end{aligned}$$

where $\partial_{U \otimes U}$ denotes the derivative in the τ direction with respect to the rank one matrix $U \otimes U$. To compute this derivative, we will take the derivatives of $r_2(\tau) + r_3(\tau) - r_1(\tau) \equiv 0$. Since this is identically zero, also the derivative is zero, hence

$$\partial_{U \otimes U} r_2(\tau_0) + \partial_{U \otimes U} r_3(\tau_0) = \partial_{U \otimes U} r_1(\tau_0) = \frac{r_1(\tau_0)}{\theta(\tau_0)} \partial_{U \otimes U} \theta(\tau_0).$$

We now note that the theta function vanishes identically on C_0 since

$$\Theta_{\tau_0} = C_0 - C_0 \supset C_0 - x_0 = C_0 - 0 = C_0.$$

So we have $\theta(\tau_0, x) = 0$ for any point $x \in C_0$. Using the heat equation again, we then see that

$$\partial_{U \otimes U} \theta(\tau_0) = \partial_U \partial_U \theta(\tau_0, z)|_{z=0} = 0$$

is zero as the derivative of the zero function (recall that U is the tangent vector to $C_0 \subset J(C_0)$ at zero). \square

In fact this statement implies the conjectural description of the strict transform of the correspondence on $C_0 \times C_0$ in genus 3:

Theorem 10. *The conjecture of Farkas and Verra holds in genus 3, i.e. the limit of the Scorza correspondence within $C_0 \times C_0$, where C_0 is*

a hyperelliptic curve of genus 3 with involution σ , is the union of the curve $\{(x, \sigma(x)) \mid x \in C_0\}$ and the diagonal with multiplicity 2.

Proof. Indeed, note that the image of the locus $\{(x, \sigma(x)) \mid x \in C_0\}$ in the Jacobian $J(C_0)$, being equal to $S_{\pi_0} = \{2x \mid x \in C_0\} \subset C_0 - C_0$, is a curve that passes through the singular point $0 \in C_0 - C_0$ with multiplicity 2. Thus its preimage on $C_0 \times C_0$ contains the two components as claimed, and it remains to show that there are no extra components.

This can be done by an intersection number computation. Indeed, for a very general curve C (and for a very general hyperelliptic curve C_0) the Neron-Severi group $NS(\text{Sym}^2 C)$ is generated by two classes: the fiber $f := \{(x, p) \mid \forall x \in C\}$ and the diagonal $\delta := \{(x, x) \mid \forall x \in C\}$. For a non-hyperelliptic curve C the Scorza correspondence is the locus $s := \{(x, y) \mid \theta(x - y) = 0\}$. It has intersection numbers $s \cdot \delta = 0$ (since the intersection is empty) and $s \cdot f = 3$ (since the degree of the theta function on the curve is equal to g , which is 3 for our case). Now for a hyperelliptic curve C_0 on $\text{Sym}^2 C_0$ we let $h := \{(x, \sigma(x)) \mid \forall x \in C_0\}$ and compute $h \cdot \delta = 8$ (since these are the fixed points of the involution) and $h \cdot f = 1$. Since we also have $\delta^2 = 2 - 2 \cdot 3 = -4$ and $\delta \cdot f = 1$, we get

$$(h + 2\delta) \cdot \delta = 8 - 2 \cdot 4 = 0 \quad \text{and} \quad (h + 2\delta) \cdot f = 1 + 2 \cdot 1 = 3.$$

It thus follows that if the strict transform of s were to contain any curve besides $h + 2\delta$, the intersection of this curve with both δ and f would be zero. Thus the limit of the Scorza correspondence over a hyperelliptic curve contains no further components, and is as claimed. \square

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